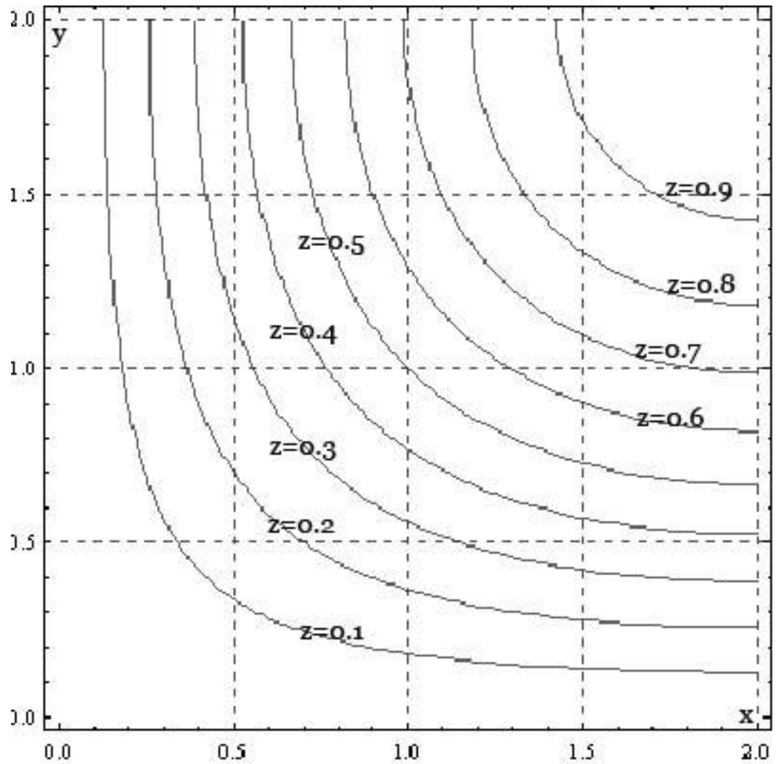
Math 2A – Vector Calculus – Chapter 11 Test – Fall '07 Name___________________________
Show your work. Don't use a calculator. Write responses on separate paper.

1. Consider the nice, smooth function $z = f(x, y)$ whose contour map is shown at right.




   a. Estimate function values to the nearest tenth to fill in the blank cells in the table below:

| $x \backslash y$ | 0.4 | 1.0 | 1.5 |
|---|---|---|---|
| 0.2 | 0.05 | | 0.14 |
| 0.4 | | | |
| 0.6 | 0.14 | | 0.42 |

   b. Use the value in your table above to estimate $f_x(0.4,1.0)$ and $f_y(0.4,1.0)$ to the nearest tenth.

   c. Let $\vec{v}$ be the vector from $P(0.6,0.4)$ to $Q(0.2,1.5)$
   Compute $D_{\vec{v}} f(0.4,1.0)$ in two ways:
   as $\frac{\Delta z}{h}$ and as $\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|}$

   d. Let $\vec{v}$ be the vector from $P(0.2,0.4)$ to $Q(0.6,1.5)$
   Compute $D_{\vec{v}} f(0.4,1.0)$ in two ways: as $\frac{\Delta z}{h}$ and as $\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|}$.

2. By considering different lines of approach, show that the function $f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$ has no limit as $(x, y) \to (0,0)$ .

3. Use polar coordinates to prove that the limit exists: $\lim_{x \to \infty} \frac{1 - e^{-x^2 - y^2}}{x^2 + y^2}$

4. Sketch level curves $f(x,y) = 0$, $f(x,y) = ½$ and $f(x,y) = – ½$ for the function $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases}$.

5. Find points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the *xy*-plane.

6. Find an equation for the plane tangent to the parametrically defined surface $\langle x, y, z \rangle = \langle \cos \ v \ , \cos \ u \ \sin \ v \ , \cos \ u \ \rangle$ where $(u,v) = (0,\pi/2)$.

7. Show that $f \ \ x, y \ = \arctan \ \ y/x$ satisfies the two dimensional Laplace equation, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

8. Find the direction in which *f* increases and decreases most rapidly at $P_0$ and the rates at which *f* changes in these directions.
   a. $f(x,y) = x^2 + \cos xy$ , $P_0(1,0)$.

   b. $f(x,y,z) = z \ln(x^2 + y^2 - 1)$ , $P_0(1,1,1)$

9. Consider $f \ \ x, y \ = x^3 + 3xy + y^3$
   a. Find the critical points.
   b. Find all maxima, minima and saddle points and evaluate the function at those points.

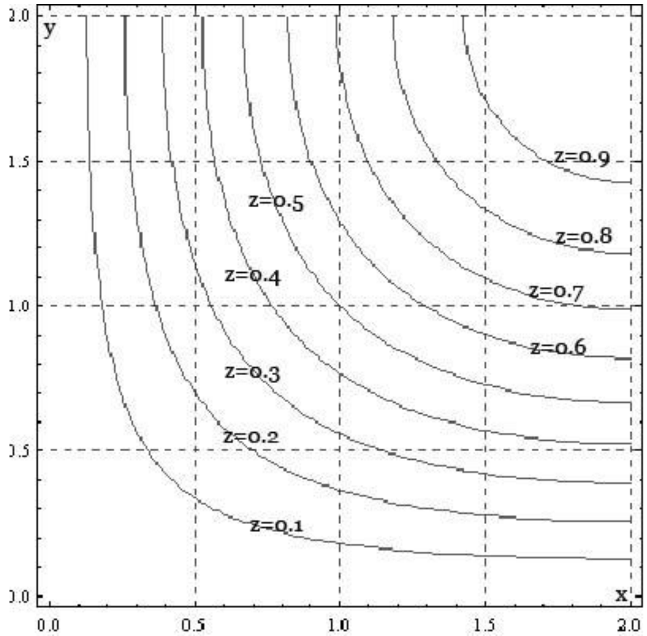
10. A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so the temperature at any point $(x, y)$ is $T \ \ x, y \ = x^2 + 2y^2 - x$

11. Find the absolute max. and min. values of $f(x, y) = xy$ on the ellipse $x^2 + 4y^2 = 8$ in two ways:
    a. By using the parameterization $\langle x, y \rangle = \langle 2\sqrt{2}\cos t, \sqrt{2}\sin t \rangle$
    b. By using Lagrange multipliers.

12. Find a level surface for the density function $f \ \ x, y, z \ = x^2 + y^2 - z^2$ that has the tangent plane $2x + 3y - z = 3$.

## Math 2A – Vector Calculus – Chapter 11 Test Solutions – Fall ’09

1. Consider the nice, smooth function $z = f(x, y)$ whose contour map is shown at right.




a) Estimate function values to the nearest tenth to fill in the blank cells in the table below:

| $x \setminus y$ | 0.4 | 1.0 | 1.5 |
|---|---|---|---|
| 0.2 | 0.05 | 0.1 | 0.14 |
| 0.4 | 0.1 | 0.2 | 0.3 |
| 0.6 | 0.14 | 0.3 | 0.42 |

b) Use the value in your table above to estimate $f_x(0.4,1.0)$ and $f_y(0.4,1.0)$ to the nearest tenth.

$$f_x(0.4,1.0) \approx \frac{\Delta z}{\Delta x} = \frac{0.3-0.1}{0.6-0.2} = \frac{1}{2}$$

$$f_y(0.4,1.0) \approx \frac{\Delta z}{\Delta y} = \frac{0.3-0.1}{1.5-0.4} = \frac{2}{11}$$

c) Let $\vec{v}$ be the vector from $P(0.6,0.4)$ to $Q(0.2,1.5)$

Compute $D_{\vec{v}} f(0.4,1.0)$ in two ways: as $\frac{\Delta z}{h}$ and as $\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|}$

SOLN: $|\vec{v}| = \left|\overrightarrow{PQ}\right| = \left|\langle 0.2-0.6, 1.5-0.4\rangle\right| = \sqrt{0.16+1.21} = \sqrt{1.37} \approx 1.17$

So, $D_{\vec{v}} f(0.4,1.0) \approx \frac{\Delta z}{h} = \frac{0}{h} = 0$ or

$$\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|} = \langle f_x, f_y \rangle \bullet \frac{\langle -0.4, 1.1\rangle}{1.17} = \langle 0.5, 0.18\rangle \bullet \frac{\langle -0.4, 1.1\rangle}{1.17} \approx \frac{-0.20+0.20}{1.17} = 0$$

d) Let $\vec{v}$ be the vector from $P(0.2,0.4)$ to $Q(0.6,1.5)$

Compute $D_{\vec{v}} f(0.4,1.0)$ in two ways: as $\frac{\Delta z}{h}$ and as $\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|}$.

SOLN: $|\vec{v}| = \left|\overrightarrow{PQ}\right| = \left|\langle 0.6-0.2, 1.5-0.4\rangle\right| = \sqrt{0.16+1.21} = \sqrt{1.37} \approx 1.17$

So, $D_{\vec{v}} f(0.4,1.0) \approx \frac{\Delta z}{h} = \frac{0.37}{1.17} \approx 0.32$ or

$$\overrightarrow{\nabla z} \bullet \frac{\vec{v}}{|\vec{v}|} = \langle f_x, f_y \rangle \bullet \frac{\langle 0.4, 1.1\rangle}{1.17} = \langle 0.5, 0.18\rangle \bullet \frac{\langle 0.4, 1.1\rangle}{1.17} \approx \frac{0.40}{1.17} \approx 0.34$$

2. By considering different lines of approach, show that the function $f(x, y) = \frac{x}{\sqrt{x^2+y^2}}$ has no limit as $(x, y) \to (0,0)$.

SOLN: Along the line $y = x$, $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\frac{x}{\sqrt{x^2+x^2}} = \lim_{x\to 0}\frac{x}{\sqrt{2}|x|}$ does not exist because

$$\lim_{x\to 0^+}\frac{x}{\sqrt{2}|x|} = \lim_{x\to 0^+}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq \lim_{x\to 0^-}\frac{x}{\sqrt{2}|x|} = \lim_{x\to 0^+}\frac{-1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

3. Use polar coordinates to prove that the limit exists: $\lim_{x\to\infty}\frac{1-e^{-x^2-y^2}}{x^2+y^2}$

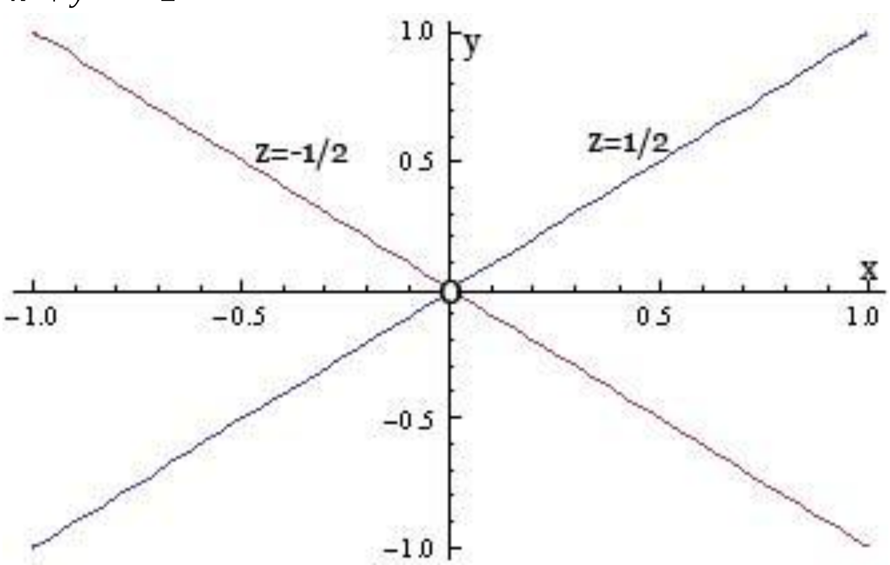
   SOLN: Since $x$ goes to infinity only if $r$ goes to infinity, $\lim_{x\to\infty}\frac{1-e^{-x^2-y^2}}{x^2+y^2} = \lim_{r\to\infty}\frac{1-e^{-r^2}}{r^2} = \frac{1-0}{\infty} = 0$

   Now it may happen that $r$ goes to infinity but, $x = 0$, but this wouldn't apply to this limit.

4. Sketch level curves $f(x,y) = 0$, $f(x,y) = ½$ and $f(x,y) = – ½$ for the function $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   SOLN: $\frac{xy}{x^2+y^2} = \frac{1}{2} \Leftrightarrow x^2 - 2xy + y^2 = 0 \Leftrightarrow (x-y)^2 = 0 \Leftrightarrow y = x$

   $\frac{xy}{x^2+y^2} = -\frac{1}{2} \Leftrightarrow x^2 + 2xy + y^2 = 0 \Leftrightarrow (x+y)^2 = 0 \Leftrightarrow y = -x$




   This shows, by the way, that there is no limiting value of $z$ as $(x,y)$ approaches (0,0).

5. Find points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the $xy$-plane.
   SOLN: We need $f_x = y + z - 1 = 0$ and $f_y = x + z = 0$. Each of these equations describes a plane, and the intersection of these planes is the line $\vec{r}(t) = \langle -t, 1-t, t\rangle = \langle 0,1,0\rangle - t\langle 1,1,-1\rangle$.

So since $y(x+z)=0$ the intersection of the line with the level surface occurs where $zx-x-z^2=0$ or $-t^2+t-t^2=-t(t-2)=0$ so either $t=0$ or $t=½$ whence the points where tangent plane is horizontal are (0,1,0) and (–½, ½, ½). To help visualize what's going on here, you might solve the equation for the surface for $z$:

$$z^2-(x+y)z+x(1-y)=0 \Leftrightarrow z=\frac{x+y\pm\sqrt{(x+y)^2-4x(1-y)}}{2}.$$ We can visualize this in Mathematica with the following command:

```
Plot3D[{{(x+y+Sqrt[(x+y)^2-4x(1-y)])/2},
{(x+y-Sqrt[(x+y)^2-4x(1-y)])/2},
{0},{1/2}},{x,-5,5},{y,-5,5}]
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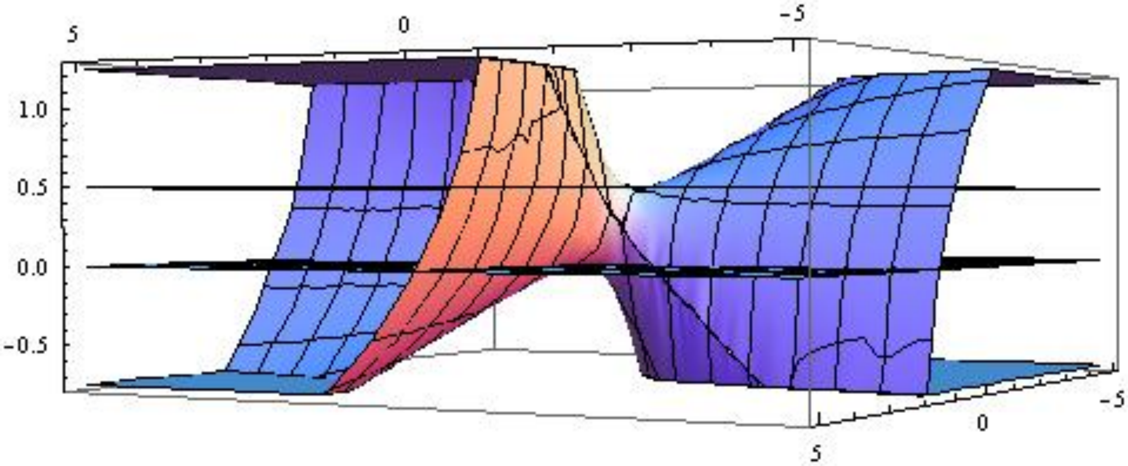



The graph shows a tilted hyperboloid of one sheet.

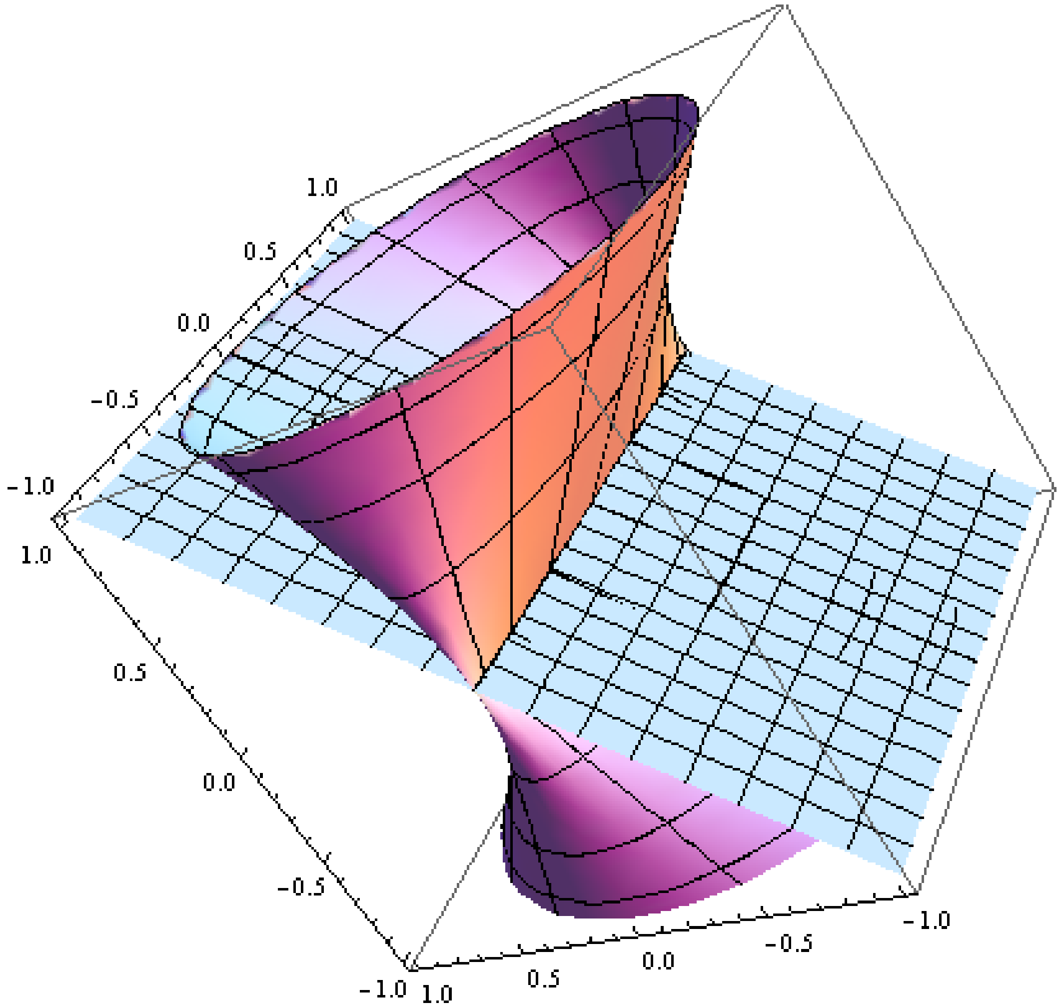
6. Find an equation for the plane tangent to the parametrically defined surface $\langle x,y,z\rangle=\langle\cos(v),\cos(u)\sin(v),\cos(u)\rangle$ where $(u,v)=(0,\pi/2)$.

SOLN: The point of tangency is $\langle\cos(\pi/2),\cos(0)\sin(\pi/2),\cos(0)\rangle=\langle 0,1,1\rangle$

The tangent vector $\vec{r}_v\,'(0,\pi/2)=\langle-\sin(\pi/2),\cos(0)\cos(\pi/2),0\rangle=\langle-1,0,0\rangle$ but the other one is $\vec{r}_u\,'(0,\pi/2)=\langle 0,-\sin(0)\sin(\pi/2),-\sin(0)\rangle=\vec{0}$, so what to do? What does this mean? It could mean there's a cusp in that direction, but this doesn't mean there isn't a tangent plane. It could be we could do a directional derivative in some other direction and get another vector for the cross product normal to the tangent plane, but the simplicity of the formula for the surface suggests we try to eliminate the parameters and get a rectangular form for the equation of the surface. Observing that $z=\cos(u)$ so $y=z\sin(v)$ leads to $y^2=z^2\sin^2(v)=z^2(1-\cos^2 v)=z^2(1-x^2)$ so the solution set to this rectangular equation can be viewed from a higher dimension as the level surface $w=0$ for the density function $w=f(x,y,z)=y^2-z^2(1-x^2)$. Thus a normal to the tangent plane at (0,1,1) can be found by evaluating the gradient vector there: $\nabla w=\left.\langle 2xz^2,2y,-2z(1-x^2)\rangle\right|_{\langle 0,1,1\rangle}=\langle 0,2,-2\rangle$ so an equation for the tangent plane is obtained by requiring that the normal be perpendicular to an arbitrary vector in the plane: $\langle 0,2,-2\rangle\bullet\langle x-0,y-1,z-1\rangle=2y-2z=0\Leftrightarrow\boxed{z=y}$

To visualize this in Mathematica, the following command will graph the surface and tangent plane:

$ParametricPlot3D[\{\{Cos[v], Cos[u]Sin[v], Cos[u]\}, \{v/Pi, u/Pi, u/Pi\}\}, \{u, -Pi, Pi\}, \{v, -Pi, Pi\}]$




It's worth examining this figure in detail. For instance, $u = 0$ is a circle of radius 1 in the plane z = 1.

7. Show that $f(x,y) = \arctan(y/x)$ satisfies the two dimensional Laplace equation, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

SOLN:

$$\nabla f(x,y) = \left\langle \frac{d}{dx}\arctan(y/x), \frac{d}{dy}\arctan(y/x) \right\rangle = \left\langle \frac{-y}{x^2\left(1+(y/x)^2\right)}, \frac{1}{x\left(1+(y/x)^2\right)} \right\rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

$$\nabla^2 f(x,y) = f_{xx} + f_{yy} = \frac{-2xy}{(x^2+y^2)^2} + \frac{2yx}{(x^2+y^2)^2} = 0$$

8. Find the direction in which $f$ increases and decreases most rapidly at $P_0$ and the rates at which $f$ changes in these directions.

a) $f(x,y) = x^2 + \cos xy$, $P_0(1,0)$.

SOLN: $\vec{\nabla} f(1,0) = \langle 2x - y\sin xy, -x\sin xy \rangle\big|_{(1,0)} = \langle 2,0 \rangle$ so $f$ grows at a rate of 2/1 in that direction.

b) $f(x,y,z) = z\ln(x^2 + y^2 - 1)$, $P_0(1,1,1)$

SOLN: $\overrightarrow{\nabla f}(1,1,1) = \left\langle \frac{2xz}{x^2+y^2-1}, \frac{2yz}{x^2+y^2-1}, \ln(x^2+y^2-1) \right\rangle\Big|_{1,1,1} = \langle 2,2,0 \rangle$

So $f$ is growing at a rate of $2\sqrt{2}/1$ is that direction.

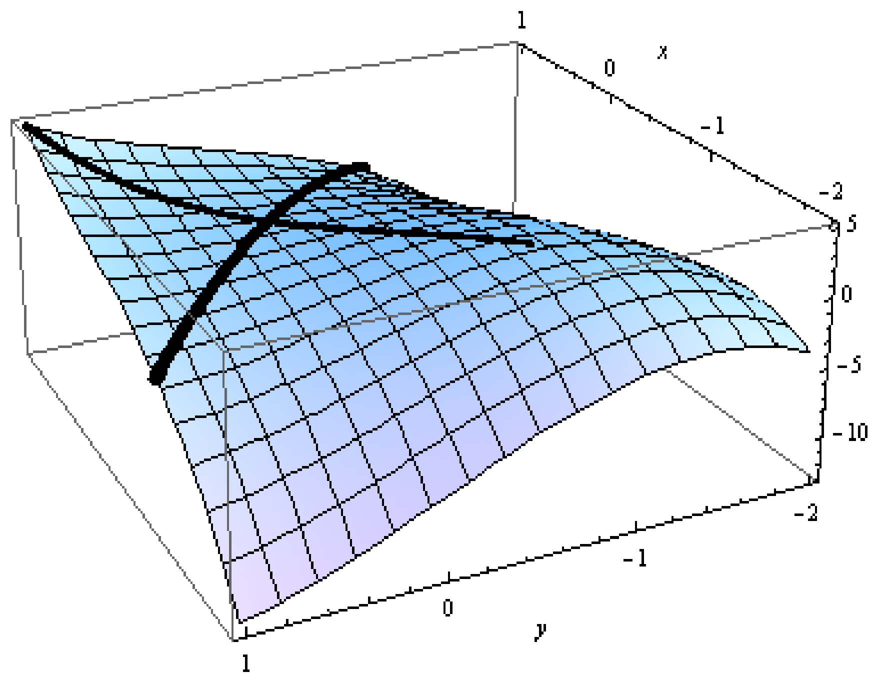
9. Consider $f(x,y) = x^3 + 3xy + y^3$

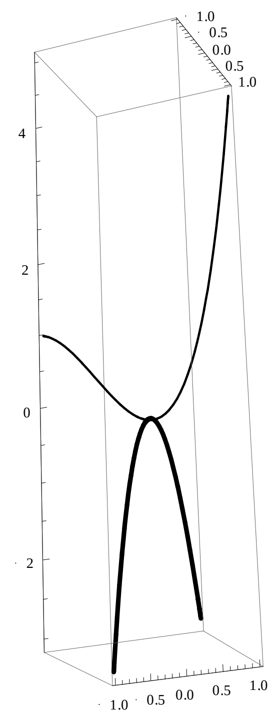
a) Find the critical points.

SOLN: $f_x = 3x^2 + 3y = 0 \Leftrightarrow y = -x^2$ and $f_y = 3x + 3y^2 = 0 \Leftrightarrow x = -y^2$, substituting, we get $y = -(-y^2)^2 = -y^4 \Leftrightarrow y = 0$ or $y = -1$, leading to critical points (0,0) and (–1, –1).

b) Find all maxima, minima and saddle points and evaluate the function at those points.

SOLN: $D = f_{xx} f_{yy} - (f_{xy})^2 = (6x)(6y) - 3^2 = 36xy - 9$ is positive at (–1, –1) where $f_{xx} = -6 < 0$, so this is a local maximum. The point (0,0) has $D < 0$ so the point is a saddle.
In the diagram below, the local max at (-1,-1,1) seems evident. The saddle is a little more subtle. Look at the curves $\vec{r}_1 = \langle t, t, 2t^3 + 3t^2 \rangle$ and $\vec{r}_2 = \langle t, -t, -3t^2 \rangle$ are shown on the plot and you can see the first is curving up at (0,0,0) and the other is curving down.



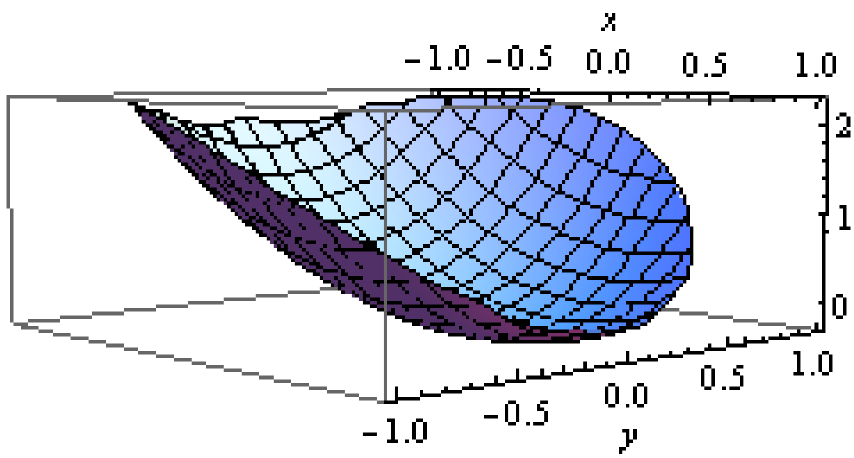

10. A flat circular plate has the shape of the region $x^2 + y^2 \le 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so the temperature at any point $(x, y)$ is $T(x,y) = x^2 + 2y^2 - x$. Find the extreme temperatures of the plate and where they occur.

SOLN: The critical points are where $T_x = 2x - 1 = 0$ and $T_y = 4y = 0$. So there's a critical point at (½, 0) where $f_{xx} = 2$ indicates that (½, 0, -¼) is a local (turns out to be global) minimum. On the boundary we have the curve

$\vec{r} = \left\langle \cos t, \sin t, \cos^2 t + 2\sin^2 t - \cos t \right\rangle = \left\langle \cos t, \sin t, 1 + \sin^2 t - \cos t \right\rangle = \left\langle \cos t, \sin t, 2 - \cos t - \cos^2 t \right\rangle$

the tangent line is horizontal when $\frac{dz}{dt} = \sin t + 2\cos t \sin t = \sin t \;\; 1 + 2\cos t \;\; = 0 \Leftarrow t = k\pi$ or $t = \pm\frac{2\pi}{3}$.

We examine the point where $t = 0$ and find (1,0,0) where $\frac{d^2 z}{dt^2} = \cos t + 2\cos 2t = 3$ and so that's a local min on the edge, (front right of edge in the image below,) but neither a max nor a min. Where $t = \pi$, (–1,0,2) has $\frac{d^2 z}{dt^2} = 1$, so that's also a local min along the edge, but neither a local min nor a local max on the surface. At $t = 2\pi/3$ the point $(-½, \sqrt{3}/2, 9/4)$ and at $t = -2\pi/3$ the point $(-½, -\sqrt{3}/2, 9/4)$ have $\frac{d^2 z}{dt^2} = -\frac{1}{2}$, so those are global maxima.




11. Find the absolute max. and min. values of $f(x, y) = xy$ on the ellipse $x^2 + 4y^2 = 8$ in two ways

a) By using the parameterization $\langle x, y\rangle = \left\langle 2\sqrt{2}\cos t, \sqrt{2}\sin t \right\rangle$

SOLN: Along the path $\langle x, y\rangle = \left\langle 2\sqrt{2}\cos t, \sqrt{2}\sin t \right\rangle$, $z = 4\cos t \sin t = 2\sin 2t$ so $z' = 4\cos 2t = 0$ if $t$ = an odd multiple of $\pi/4$. At $\pi/4$ and $5\pi/4$ $z'' = -4$ so (2, 1, 2) and (–2,–1,2) are global maxima and at $3\pi/4$ and $7\pi/4$ $z'' = 4$ so (–2, 1, –2) and (2,–1, –2) are global minima.

b) By using Lagrange multipliers.

SOLN: $\nabla f = \lambda \nabla g \Leftrightarrow \langle y, x\rangle = \lambda \langle 2x, 8y\rangle$ leads to the system

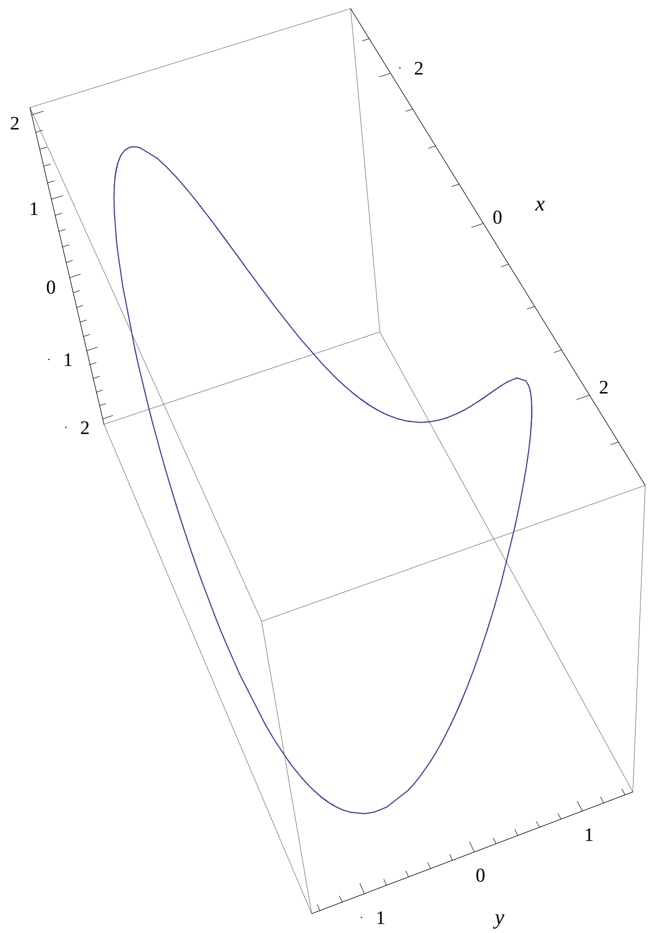
$y = 2\lambda x$

$x = 8\lambda y$

$x^2 + 4y^2 = 8$

Substituting from the second to the first, $y = 16\lambda^2 y$, we know that either $y = 0$ or $\lambda = \pm ¼$. If $y = 0$ then $x = 0$ and then constraint $x^2 + 4y^2 = 8$ can't be met so $\lambda = \pm ¼$ which means $y = \pm x/2$ and substituting into the the ellipse equation, $x^2 + x^2 = 8 \Leftrightarrow x = \pm 2$ meaning that $y = \mp 1$

After investigating we determine that the global max is 2 occurring at (2,1) and (–2,–1) and the global min is –2 occurring at (–2,1) and (2,–1).




12. Find a level surface for the density function $f(x, y, z) = x^2 + y^2 - z^2$ that has the tangent plane $2x + 3y - z = 3$.

SOLN: The normal to the level surface will be parallel to the normal to the plane if $\overrightarrow{\nabla f} = \langle 2x, 2y, -2z \rangle = \lambda \langle 2, 3, -1 \rangle$ so that $\lambda = x = 2y/3 = 2z$ and substituting into the equation of the plane, $2\lambda + 9\lambda/2 - \lambda/2 = 6\lambda = 3$ or $\lambda = 1/2$ and thus

$f(1/2, 3/4, 1/4) = (1/2)^2 + (3/4)^2 - (1/4)^2 = 3/4$. So the level surface is $x^2 + y^2 - z^2 = 3/4$